Cascaded LQR-Based Control of a Micro Quadrotor

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Introduction

Quadrotors are underactuated aerial vehicles with 6 DOF and 4 independent thrust-producing motors.

Control Objectives:

- **Step Reference (new hover position):**
  - Attitude: $t_s < 1$ s, $e_{ss} = 0$ deg
  - Position: $t_s < 5$ s, $e_{ss} = 0$ m
- **Ramp Reference (linear trajectory):**
  - Position: $e_{ss} = 0$ m
- **Sinusoidal Reference (oscillatory trajectory):**
  - Position: $e_{\text{peak}} < 0.05$ m
  - Phase Lag: $\varphi < 15°$ @ *1 rad/s*

Nonlinear Equations of Motion

The outer loop (position) system is governed by the following, where $\ddot{\mathbf{p}} = [x, y, z]^T$ is in the inertial frame.

$$m\ddot{\mathbf{p}} = mg\vec{z} + {}^{I}R_{B}{}^{B}\mathbf{F} \tag{1}$$

Expressed in the body frame, the acceleration is as follows:

$${}^{B}\ddot{\mathbf{a}} = \frac{{}^{B}\mathbf{F}}{m} + {}^{B}R_{I}g\vec{z} - \omega \times {}^{B}\mathbf{v} \tag{2}$$

The inner loop (attitude) system is governed by the following, where $\boldsymbol{\omega} = [p, q, r]^T$ is in the body frame.

$$\mathbf{I}\dot{\omega} = -\omega \times \mathbf{I}\omega + \boldsymbol{\tau} \tag{3}$$

Linearized Equations of Motion, Position

Linearized about a hover condition at $[0, 0, 0]^T$, the position subsystems result in (using Equations (1) and (2)):

$$^{B}a_x = -\theta g \tag{4}$$

$$^{B}a_y = \phi g \tag{5}$$

$$\ddot{z} = \frac{1}{m}(mg - T) \tag{6}$$

Linearization Assumptions:

- First-order Taylor series expansion.
- $\sin(\theta) \approx \theta, \ \cos(\theta) \approx 1$.
- Neglecting higher-order terms: $O(\delta^2)$.

Linearized Equations of Motion, Attitude

Linearized about a hover condition at $[0, 0, 0]^T$, the attitude subsystems result in (using Equation (3)):

$$\ddot{\phi} = \frac{\tau_\phi}{I_{xx}} \tag{7}$$

$$\ddot{\theta} = \frac{\tau_\theta}{I_{yy}} \tag{8}$$

$$\ddot{\psi} = \frac{\tau_\phi}{I_{zz}} \tag{9}$$

Linearization Assumptions:

- First-order Taylor series expansion.
- $\sin(\theta) \approx \theta, \ \cos(\theta) \approx 1.$
- Neglecting higher-order terms: $\mathcal{O}(\delta^2)$.

Open-Loop State Space Models

Outer State-Space Model

$$\begin{bmatrix} {}^B\dot{x}_I \\ {}^B\ddot{x}_I \\ {}^B\dot{y}_I \\ {}^B\ddot{y}_I \\ \dot{z} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} {}^Bx_I \\ {}^B\dot{x}_I \\ {}^By_I \\ {}^B\dot{y}_I \\ z \\ \dot{z} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -g & 0 & 0 \\ 0 & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{m} \end{bmatrix} \begin{bmatrix} \theta \\ \phi \\ mg - T \end{bmatrix} \tag{10}$$

Inner State-Space Model

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{\phi} \\ \ddot{\phi} \\ \dot{\psi} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \\ \psi \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{I_{yy}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{I_{xx}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{I_{zz}} \end{bmatrix} \begin{bmatrix} \tau_\theta \\ \tau_\phi \\ \tau_\psi \end{bmatrix} \tag{11}$$

Observability and Controllability

Investigating controllability and observability:

$$\operatorname{rank}\left(\mathcal{C}^{\text{outer}}\right) = n_{\text{outer}}, \quad \operatorname{rank}\left(\mathcal{O}^{\text{outer}}\right) = n_{\text{outer}}$$
$$\operatorname{rank}\left(\mathcal{C}^{\text{inner}}\right) = n_{\text{inner}}, \quad \operatorname{rank}\left(\mathcal{O}^{\text{inner}}\right) = n_{\text{inner}}$$

where $\mathcal{C}$ and $\mathcal{O}$ are the controllability and observability Gramians, respectively, and n is the number of states.

Therefore, each system is both **controllable** and **observable**.

LQR Control Structure

Assuming full-state feedback (FSFB) for both the inner and outer loops:

$$\mathbf{u}_{\text{outer}} = -\mathbf{K}_{\text{outer}}\mathbf{x}_{\text{outer}} = \begin{bmatrix} \theta_{\text{des}} & \phi_{\text{des}} & mg - T \end{bmatrix}$$

$$\mathbf{u}_{\text{inner}} = -\mathbf{K}_{\text{inner}}\mathbf{x}_{\text{inner}} = \begin{bmatrix} \tau_{\theta} & \tau_{\phi} & \tau_{\psi} \end{bmatrix}$$

The standard LQR design (without integral action) minimizes the following cost function:

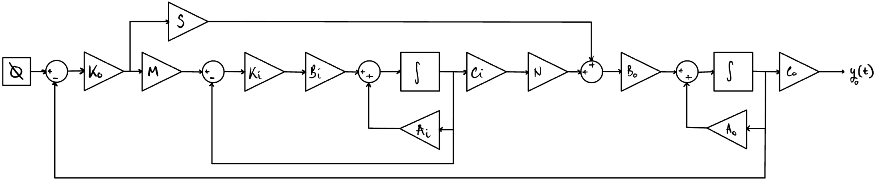
$$J = \int_0^\infty \left(\mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{u}^T \mathbf{R} \mathbf{u} \right) dt$$

The weighting matrices were defined as:

$$\begin{aligned}
\mathbf{Q}_{\text{outer}} &= \text{diag}(25,\ 5,\ 25,\ 5,\ 50,\ 10), \\
\mathbf{R}_{\text{outer}} &= \text{diag}(1000,\ 1000,\ 100), \\
\mathbf{Q}_{\text{inner}} &= \text{diag}(5,\ 0.1,\ 5,\ 0.1,\ 2.5,\ 0.05), \\
\mathbf{R}_{\text{inner}} &= \text{diag}(30,\ 30,\ 3000).
\end{aligned}$$

Proving Closed-Loop Stability

With a cascaded control structure, how can you prove stability? For a desired hover condition ($\mathbf{x}_{\text{des}} = 0 \in \mathbb{R}^{6\times 1}$), and ignoring frequency differences:



Using block diagram algebra, the following was derived:

$$\begin{bmatrix} \mathbf{x}_\text{o} \\ \mathbf{x}_\text{i} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_\text{o} - \mathbf{B}_\text{o}\mathbf{S}\mathbf{K}_\text{o} & \mathbf{B}_\text{o}\mathbf{N}\mathbf{C}_\text{i} \\ -\mathbf{B}_\text{i}\mathbf{K}_\text{i}\mathbf{M}\mathbf{K}_\text{o} & \mathbf{A}_\text{i} - \mathbf{B}_\text{i}\mathbf{K}_\text{i} \end{bmatrix} \begin{bmatrix} \mathbf{x}_\text{o} \\ \mathbf{x}_\text{i} \end{bmatrix} \tag{12}$$

$$\mathbf{M}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Proving Closed-Loop Stability, Cont.

Closed-Loop Poles:
The system has 12 closed-loop poles:

$$\begin{bmatrix} -995.42 & -801.86 & -40.19 & -7.18 \\ -4.71 & -4.71 & -3.09 + 0.39i & -3.09 - 0.39i \\ -1.17 + 0.98i & -1.17 - 0.98i & -1.17 + 0.98i & -1.17 - 0.98i \end{bmatrix}$$

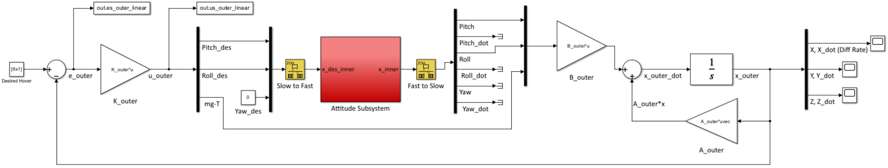
All poles have strictly negative real parts $\Rightarrow$ **asymptotically stable**.

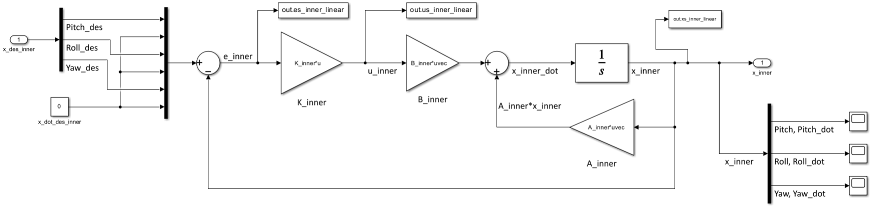
Estimated Settling Times:

- x and y position dynamics ($-1.17 \pm 0.98i$): $t_s \approx \frac{4}{1.17} \approx 3.42$ s
- z position ($-3.09 \pm 0.39i$): $t_s \approx \frac{4}{3.09} \approx 1.29$ s
- ϕ, θ (-4.71): $t_s \approx \frac{4}{4.71} \approx 0.85$ s
- ψ (-7.18): $t_s \approx \frac{4}{7.18} \approx 0.56$ s

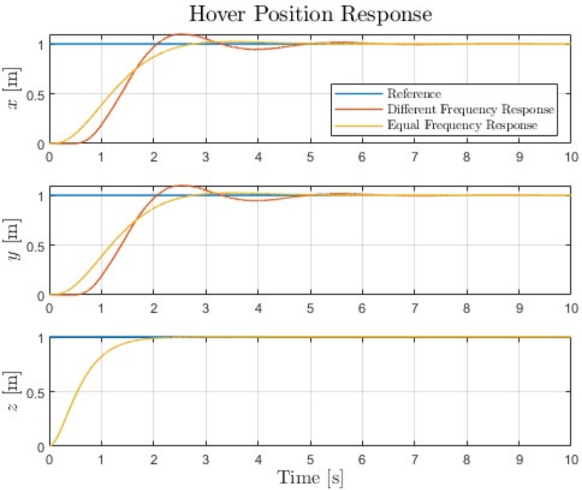
Meeting Step Response Requirements!

The Linear Simulink Model

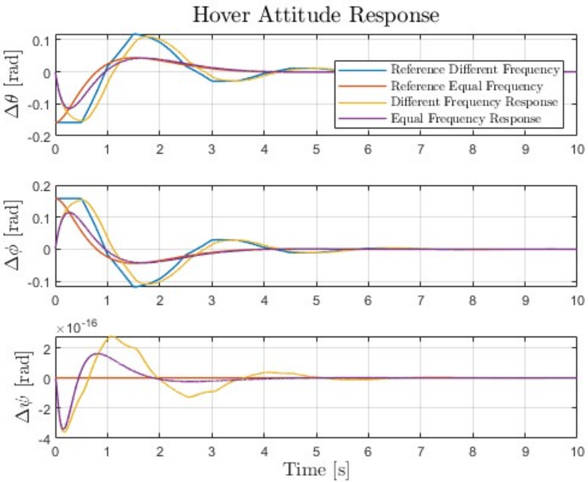




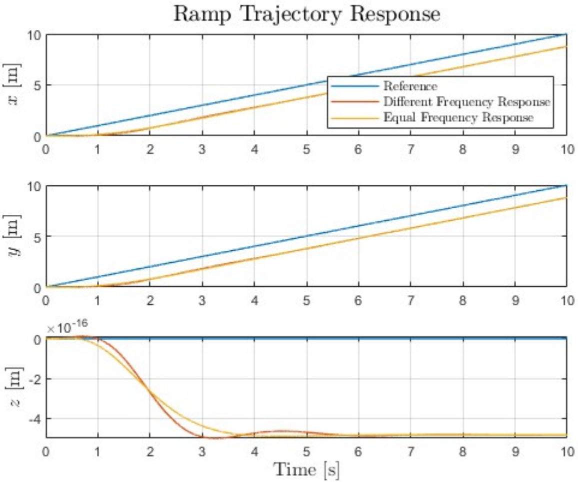
Desired Hover (Step) Position Response



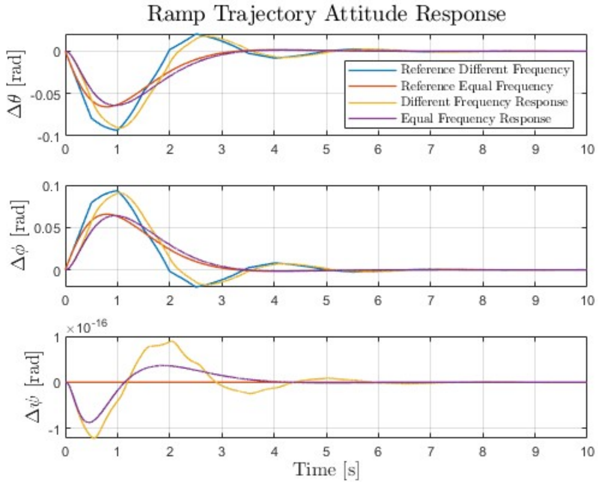
Desired Hover (Step) Attitude Response



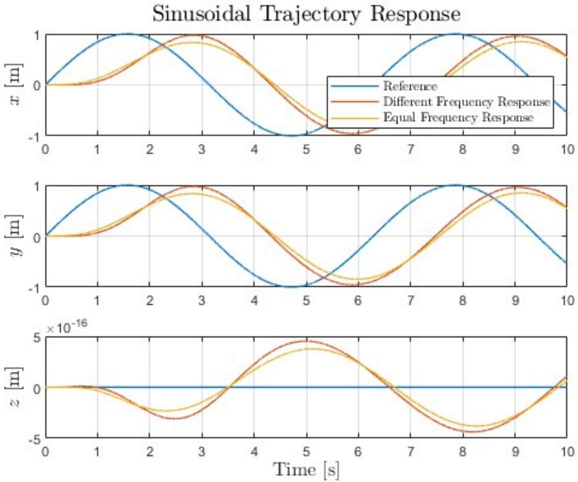
Ramp Trajectory Position Response



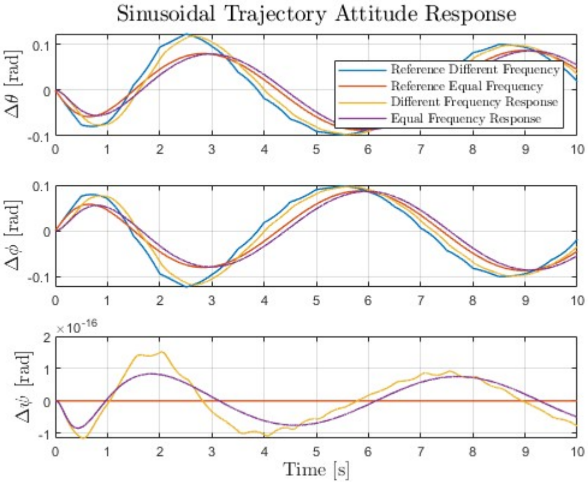
Ramp Trajectory Attitude Response



Sinusoidal Trajectory Position Response



Sinusoidal Trajectory Attitude Response



Future Work Outline

1. **Add Integral Action**

$$\mathbf{u}_t = -\mathbf{K}_x \mathbf{x}_t - \mathbf{K}_i \int_0^t \mathbf{e}_\tau \, d\tau$$

2. **Develop Full Nonlinear Model**
   Extend system modeling beyond linearization to capture higher-order coupling and actuator limits.

3. **Train Neural Network on Residual Dynamics**

$$\mathbf{x}_{t+1} = \underbrace{f(\mathbf{x}_t, \mathbf{u}_t)}_{\text{linear model}} + \underbrace{g_t}_{\text{nonlinear effects}} + \underbrace{h_t}_{\text{residual dynamics}}$$

$$\mathbf{u}_t = \underbrace{\pi_{\mathrm{LQR}}(\mathbf{e}_t)}_{\text{nominal controller}} \underbrace{+\pi_l(\mathbf{e}_t, \mathbf{u}_{\mathrm{LQR},t})}_{\text{NN correction}}$$

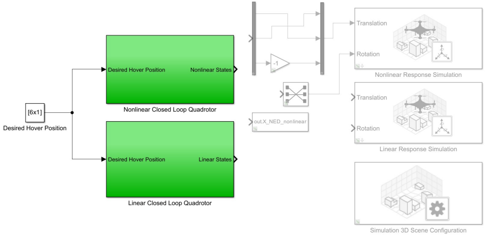
4. **Prove NN Stability Using Lipschitz Constant**

$$\|\mathcal{N}(x_1) - \mathcal{N}(x_2)\| \le L\|x_1 - x_2\|, \quad \forall x_1, x_2$$

   Bound residual policy behavior and incorporate into Lyapunov-based analysis.

Future Work Status

1. **Add Integral Action:** Not Started
2. **Develop Full Nonlinear Model:** In Progress



3. **Train Neural Network on Residual Dynamics:** Loss Function Developed

$$\mathcal{L} = \sum_{t=0}^{T} \left\| \mathbf{x}_t^{\text{nonlinear}} - \mathbf{x}_t^{\text{linear}} \right\|^2$$

4. **Prove NN Stability Using Lipschitz Constant:** Not Started

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